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MID-SPAN DEFLECTION AND END-SHORTENING OF A ROD AFTER BUCKLING

A. V. Anfilof'ev

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The load–displacement relations governing the postbuckling behavior are expressed in terms of elementary functions. An approximate solution of the elastica problem with modified expressions for the curvature is given. Equations of the elastic curve are obtained with the use of an approximate determination of elliptic integrals.

In studying the postbuckling behavior of a rod, the load–deflection relation is of interest. Birger, Shorr, and Iosilevich [1] proposed the approximate relation

$$\frac{f}{L} \approx \frac{2\sqrt{2}}{\pi}\sqrt{1 - \frac{P_{\rm E}}{P}},$$

where f is the mid-span deflection, L is the length of the rod, $P_{\rm E}$ is the Euler load, and P is the postbuckling load. Nikolai [2] stated that the relation between the load and the postbuckling deflection is completely understood; he also expanded this relation in a power series and obtained the expressions

$$\frac{f}{L} \approx \frac{2\sqrt{2}}{\pi}\sqrt{\frac{P}{P_{\rm E}} - 1}, \qquad \frac{f}{L} \approx \frac{2\sqrt{2}}{\pi}\sqrt{\frac{P}{P_{\rm E}} - 1}\Big[1 - \frac{19}{16}\Big(\frac{P}{P_{\rm E}} - 1\Big)\Big]$$

by retaining two and three terms of the expansion, respectively. Less attention has been given to the end shortening. Mises approximated the curvature of the rod by the first and third harmonics of the trigonometric series and obtained both relations [3]:

$$\frac{f}{L} \approx \frac{2\sqrt{2}}{\pi}\sqrt{\frac{P}{P_{\rm E}} - 1}\Big[1 - \frac{1}{8}\Big(\frac{P}{P_{\rm E}} - 1\Big)\Big], \qquad \frac{\Delta}{L} \approx \frac{2(P - P_{\rm E})}{2P - P_{\rm E}}$$

(Δ is the end-shortening of the rod). These formulas do not cover the entire range of curvatures of the rod and show only that the deflections increase rapidly if the load exceeds slightly the Euler load.

Retaining four terms of the power series, Astapov [4] proposed a formula for calculating the deflections that gives an error of 3% or smaller for loads exceeding the critical load by a factor of 3.5. Astapov [5] derived approximate formulas that allow one to calculate the coordinates of the elastic curve of the rod with an error smaller than 1% of the length of the rod for loads exceeding the Euler load by 30%. To determine large curvatures of the rod more accurately, one should retain more and more terms of the expansion, which leads to cumbersome functional relations. It is, therefore, of interest to solve this problem for an arbitrary curvature of the rod.

The equations of the elastica of a simply supported rod of constant cross section ($EJ = \text{const}$) under a load $p = P/(EJ)$ are usually formulated with the use of the well-known expression for the curvature of a plane curve:

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$$\frac{d\theta}{dL} = -py, \qquad \frac{d^2y}{dx^2} \Big/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = -py.$$

Here θ is the slope of the tangent to the curve $y(x)$ over its length L and the arc length L_x, which is reckoned from the coordinate origin, is taken to be an independent variable. These equations are nonlinear and their solutions have singularities [6–8].

To simplify the solution of the problem, we use expressions given in [9], in which the curvature is interpreted as a rate of variation of the trigonometric functions of the slope of the tangent over the lengths of projections of the curve. The problem is governed by the system of differential relations, in which the slope of the tangent to the curve used as an independent variable:

$$\frac{d\cos\theta}{dy} = py, \qquad -\frac{d\sin\theta}{dx} = py, \quad dL = \frac{dy}{\sin\theta}. \tag{1}$$

Given the initial conditions $\theta = \theta_0$ for $x = 0$ and $y = 0$, the first equation of system (1) implies the equation

$$y\sqrt{p} = \sqrt{2(\cos\theta - \cos\theta_0)}. \tag{2}$$

With allowance for substitution (2), the second and third equations in (1) can be combined to give

$$dx\,\sqrt{p} = -\frac{\cos\theta d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}, \qquad dL\,\sqrt{p} = -\frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}.$$

Integration from the coordinate origin to an arbitrary point of the curve (x, y, θ) yields the equation of the abscissas of the points of the curve and the arc length:

$$x\sqrt{p} = -\int\limits_{\theta_0}^{\theta} \frac{\cos\theta d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}, \qquad L_x\sqrt{p} = -\int\limits_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\theta_0)}}. \tag{3}$$

The integrals in (3) are not elementary functions. We introduce the variable k:

$$k = \sin(\theta_0/2), \qquad \sin\varphi = \sin(\theta/2)/k. \tag{4}$$

With allowance for (4) and the relation $\cos\theta = 1 - 2\sin^2(\theta/2)$, the integrals in (3) become normal elliptic integrals the values of which are tabulated as functions of the argument φ and the modulus k (or the angle $\alpha = \arcsin k$). The equations of the elastica of the rod are written in terms of elliptic integrals as follows:

$$y\sqrt{p} = 2k\cos\varphi; \tag{5}$$

$$x\sqrt{p} = \int\limits_{\pi/2}^{\varphi} \frac{d\varphi}{\sqrt{1 - (k\sin\varphi)^2}} - 2\int\limits_{\pi/2}^{\varphi} \sqrt{1 - (k\sin\varphi)^2}\,d\varphi$$

$$= F(\varphi, k) - F(\pi/2, k) - 2[E(\varphi, k) - E(\pi/2, k)]; \tag{6}$$

$$L_x\sqrt{p} = -\int\limits_{\pi/2}^{\varphi} \frac{d\varphi}{\sqrt{1 - (k\sin\varphi)^2}} = F(\pi/2, k) - F(\varphi, k). \tag{7}$$

Here $F(\varphi, k)$ and $E(\varphi, k)$ are the incomplete elliptic integrals of the first and second kind, respectively. For $\varphi = \pi/2$, these integrals are called the complete elliptic integrals.

Using the substitution $x = l/2$, $y = f$, and $\theta = 0$, from Eqs. (4)–(7) we obtain the mid-span deflection, the chord, and the length of the rod: $f\sqrt{p} = 2k$, $l\sqrt{p} = 2[2E(\pi/2, k) - F(\pi/2, k)]$, and $L\sqrt{p} = 2F(\pi/2, k)$. The dimensionless displacements are given by

$$\frac{f}{L} = \frac{k}{F(\pi/2, k)}, \qquad \frac{\Delta}{L} = 1 - \frac{l}{L} = 2\Big[1 - \frac{E(\pi/2, k)}{F(\pi/2, k)}\Big], \tag{8}$$

and the dimensionless load is

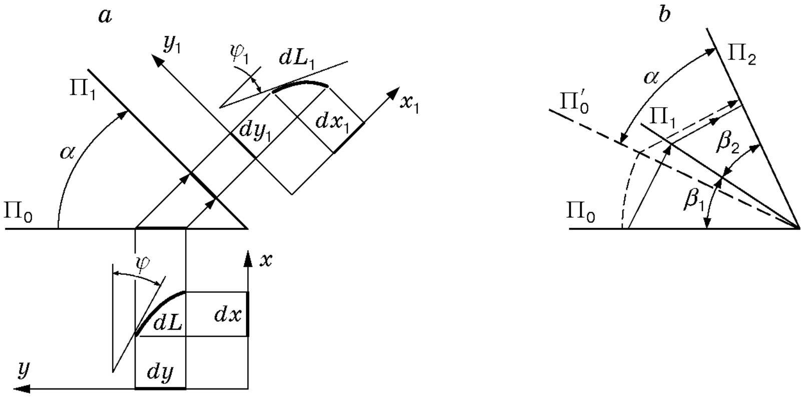

Fig. 1. Mapping of a line from the plane Π_0 onto the inclined plane Π_1 (a) and double mapping $\Pi_0 \to \Pi_1 \to \Pi_2$ and an equivalent mapping $\Pi_0' \to \Pi_2$ (b).

$$P/P_{\rm E} = [2F(\pi/2, k)/\pi]^2. \tag{9}$$

Substituting (9) into (8), we obtain

$$\frac{f}{L} = \frac{2}{\pi} k \sqrt{\frac{P_{\rm E}}{P}}, \qquad \frac{\Delta}{L} = 2\Big[1 - \sqrt{\frac{P_{\rm E}}{P}}\, \frac{2E(\pi/2, k)}{\pi}\Big]. \tag{10}$$

It follows from (10) that there are no elementary relations between the load and the displacements, since there are no elementary relations between the elliptic integrals and the modulus k. The necessary relations can be only approximate and they can be obtained by approximating the initial integrals in (3) or the corresponding elliptic integrals by elementary functions. The formal representation of these integrals by a finite number of terms of a power or trigonometric series [1, 3] is not effective, since the series converge slowly. One should resort to other approaches, for example, to seek for an approximate expression of one of the elliptic integrals and determine another by using relations between these integrals.

The integral of the first kind determines the arc lengths of the Bernoulli lemniscate, and the integral of the second kind determines the arc length of an ellipse and admits a simpler geometrical interpretation. The ellipse can be obtained by projecting the circumference onto a plane inclined at a certain angle to the circumference plane. Consequently, one can readily express the arc length of the ellipse in terms of the circumference parameters.

We consider the mapping of the curve $y = f(x)$ from the plane Π_0 onto the plane Π_1 (α is the angle between the planes) (Fig. 1a). On the plane Π_0, the projections of the differential of the arc dL onto the coordinate axes have the form $dy = dL \sin\varphi$ and $dx = dL \cos\varphi$. On the plane Π_1, we obtain the curve $y_1 = f(x_1)$ for which the differential of the arc length is given by $dL_1 = \sqrt{dy_1^2 + dx_1^2}$. Using the relations $dy_1 = dy \cos\alpha = dL \sin\varphi \cos\alpha$ and $dx_1 = dx$, we obtain

$$dL_1 = dL\sqrt{1 - \sin^2\!\alpha \sin\varphi}. \tag{11}$$

It follows from (11) that if there is a circular arc of radius R with a central angle φ on the plane Π_0, the length of its projection onto the plane Π_1 is determined by the elliptic integral of the second kind:

$$L_1 = R \int\limits_0^{\varphi} \sqrt{1 - (\sin\alpha \sin\varphi)^2}\, d\varphi = RE(\varphi, k). \tag{12}$$

According to (11), $E(\varphi, k)$ is the angular measure of the arc of a compressed ellipse with central angle φ. Here the angle α, the modulus k, and the argument φ have a specific geometrical representation. For $\alpha = 0$, the length of the circumference arc is also determined by the elliptic integral: $E(\varphi, 0) = \varphi$. Upon rotation of the plane Π_1 from 0 to $\pi/2$ and further from $\pi/2$ to π or upon backward rotation, one can readily determine the length of the ellipse arc: the integral $E(\varphi, k)$ is a periodic function of the angle α.

To obtain an approximate expression for integral (12), we perform two successively transformations according to the scheme shown in Fig. 1b. After mapping from the plane Π_0 onto the plane Π_1, in accordance with (11), we obtain $dL_1 = dL\sqrt{1-\sin^2\beta_1\sin^2\varphi}$. After the second mapping from the plane Π_1 onto the plane Π_2, we have $dL_2 = dL_1\sqrt{1-\sin^2\beta_2\sin^2\varphi_1} = dL\sqrt{1-\sin^2\beta_1\sin\varphi}\sqrt{1-\sin^2\beta_2\sin\varphi_1}$. As a result, assuming that $\sin^2\varphi = \sin^2\varphi_1$ for $\beta_1 = \beta_2 = \beta$, we obtain

$$dL_2 \approx dL(1-\sin^2\beta\sin^2\varphi). \tag{13}$$

The above assumption implies the equality $\sin^2\varphi + \cos^2\varphi/\cos^2\beta \approx 1$.

One mapping from the plane Π_0' (the rotated plane Π_0) onto the plane Π_2 with the angle α between them is equivalent to two successive mappings:

$$dL_2 = dL\sqrt{1-\sin^2\alpha\sin^2\varphi}. \tag{14}$$

According to Fig. 1b, we obtain $\cos\alpha = \cos\beta_1\cos\beta_2 = \cos^2\beta$. With allowance for this relation, expressions (13) and (14) can be combined to give

$$\sqrt{1-\sin^2\alpha\sin^2\varphi} \approx 1-(1-\cos^2\beta)\sin^2\varphi = \cos^2\varphi + \cos\alpha\sin^2\varphi. \tag{15}$$

Using the integrand (15), we obtain the approximate expression for an elliptic integral of the second kind

$$E(\varphi,k) \approx \int_0^\varphi (\cos^2\varphi + \cos\alpha\sin^2\varphi)\,d\varphi = \varphi\Big(\cos^2\frac{\alpha}{2} + \frac{\sin 2\varphi}{2\varphi}\sin^2\frac{\alpha}{2}\Big). \tag{16}$$

The formula $\partial E(\varphi,k)/\partial k = [E(\varphi,k) - F(\varphi,k)]/k$ [10] with $k = \sin\alpha$ implies the equation

$$F(\varphi,k) = E(\varphi,k) - \tan\,\alpha\,\frac{dE(\varphi,k)}{d\alpha}. \tag{17}$$

Substituting (16) into (17), we arrive at the expression for an elliptic integral of the first kind

$$F(\varphi,k) \approx \frac{\varphi}{\cos\alpha}\Big(\cos^2\frac{\alpha}{2} - \frac{\sin 2\varphi}{2\varphi}\sin^2\frac{\alpha}{2}\Big). \tag{18}$$

From (16) and (18) for $\varphi = \pi/2$, we obtain the complete integrals

$$E(\pi/2,k) \approx \frac{\pi}{2}\cos^2\frac{\alpha}{2}, \qquad F(\pi/2,k) \approx \frac{\pi}{2}\,\frac{1}{1-\tan^2(\alpha/2)}. \tag{19}$$

Using the approximate expressions for integrals (16), (18), and (19) for $\alpha = \theta_0/2$, we write the equations of an elastic curve (5) and (6) in the form

$$\frac{y}{L} \approx \frac{\sin\theta_0}{\pi\cos^2(\theta_0/4)}\cos\varphi, \quad \frac{x}{L} \approx \Big(\frac{2\varphi}{\pi}-1\Big)\Big(1-2\cos\frac{\theta_0}{2}\Big) - \frac{\sin 2\varphi}{\pi}\tan^2\frac{\theta_0}{4}\Big(1+2\cos\frac{\theta_0}{2}\Big),$$

where, according to (4), $\varphi = \arcsin[\sin(\theta/2)/\sin(\theta_0/2)]$. Using these equations, one can plot the elastic curve for a given rotation of the rod cross section in the coordinate origin θ_0. The coordinates of the points are calculated with the use of the values of the angle θ. The load that corresponds to the curve is determined from formula (9) which takes the form

$$\frac{P}{P_{\mathrm{E}}} \approx \frac{1}{[1-\tan^2(\theta_0/4)]^2} \tag{20}$$

with allowance for (19).

Figure 2 shows elastic curves of the rod whose angle θ varies in the range $-\theta_0 \leqslant \theta \leqslant \theta_0$. It is noteworthy that the ordinates of the curves are calculated more accurately than the abscissas. For example, for $\theta_0 = 90°$, the calculation errors for the integrals of the first and second kind amount to +2.3% and −0.72%, respectively, and the errors of approximate determination of the mid-span deflection and chord are −0.85% and −4.3%, respectively. The calculated error is 4.5% smaller than the exact value.

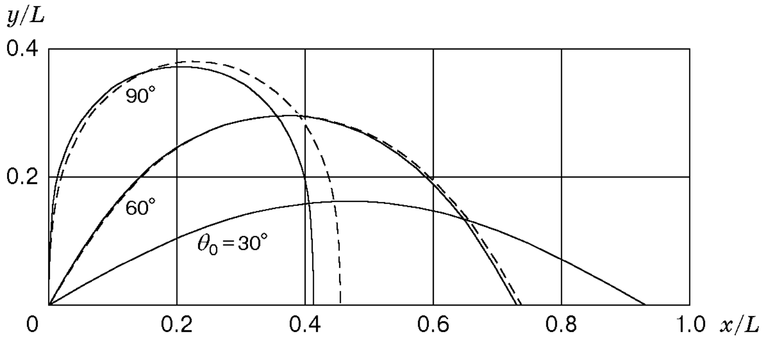

Fig. 2. Postbuckling bending of the rod: dashed curves refer to the exact elliptic-integral solution and solid curves to the approximate solution.

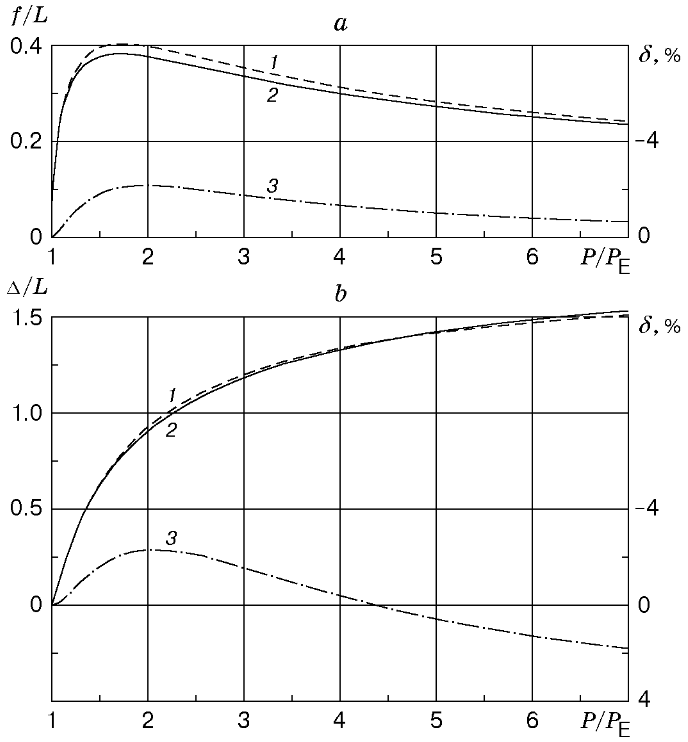

Fig. 3. Mid-span deflection (a) and end-shortening of the rod (b) versus load: curves 1 refer to the exact solution, curves 2 to the approximate solution, and curves 3 to the error δ.

To determine the elastic curve for a given load, one should find the angle $\theta_0 \approx 4\arctan\sqrt{1-\sqrt{P_{\rm E}/P}}$ from relation (20) and, then, the coordinates of the curve. For example, for the load $P/P_{\rm E} = 1.393$, we obtain $\theta_0 \approx 85.4^\circ$, whereas the exact value is $\theta_0 = 90^\circ$ (the error is 5.1%). In this case, the mid-span deflection and the chord are determined with the errors -1.6% and $+1.3\%$, respectively.

With allowance for the approximate expressions for the complete elliptic integrals (19), formulas (10) yield the mid-span deflection and the end-shortening of the rod as functions of the load

$$\frac{f}{L} \approx \frac{2}{\pi}\sqrt{\frac{P_{\rm E}}{P}\Big[1-\frac{P_{\rm E}/P}{(2-\sqrt{P_{\rm E}/P})^2}\Big]}; \qquad \frac{\Delta}{L} \approx 2\Big(1-\frac{\sqrt{P_{\rm E}/P}}{2-\sqrt{P_{\rm E}/P}}\Big). \tag{21}$$

A comparison of the calculation results obtained from formulas (21) and those obtained from formulas (8) and (9), which are represented in Fig. 3, shows that the approximate formulas proposed allow one to calculate the displacements with the use of the load with satisfactory accuracy in the entire range of rod curvatures after buckling.

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